

Yetter-Drinfeld-Long bimodules are modules

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Abstract. Let H be a finite dimensional bialgebra. In this paper, we prove that the category of Yetter-Drinfeld-Long bimodules is isomorphic to the Yetter-Drinfeld category over the tensor product bialgebra $H \otimes H^*$ as monoidal category. Moreover if H is a Hopf algebra with bijective antipode, the isomorphism is braided.

Keywords: Hopf algebra; Yetter-Drinfeld-Long bimodule; Braided monoidal category.

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Introduction

F. Panaite and F. V. Oystaeyen in [3] introduced the notion of L - R smash biproduct, with the L - R smash product and L - R smash coproduct introduced in [2] as multiplication, respectively comultiplication. When an object A which is both an algebra and a coalgebra and a bialgebra H form a L - R -admissible pair (H, A) , $A \sharp H$ becomes a bialgebra with smash product and smash coproduct, and the Radford biproduct is a special case. It turns out that A is in fact a bialgebra in the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules (introduced in [3]) with some compatible condition.

The aim of this paper is to show that the category $\mathcal{LR}(H)$ coincides with the Yetter-Drinfeld category over the bialgebra $H \otimes H^*$, in the case when H is finite dimensional. Hence any object $M \in \mathcal{LR}(H)$ is just a module over the Drinfeld double $D(H \otimes H^*)$.

The paper is organized as follows. In section 1, we recall the category $\mathcal{LR}(H)$. In section 2, we give the main result of this paper.

Throughout this article, all the vector spaces, tensor product and homomorphisms are over a fixed field k . For a coalgebra C , we will use the Heyneman-Sweedler's notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$ (summation omitted).

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1 Preliminaries

Let H be a bialgebra. The category $\mathcal{LR}(H)$ is defined as follows. The objects of $\mathcal{LR}(H)$ are vector spaces M endowed with H -bimodule and H -bicomodule structures (denoted by $h \otimes m \mapsto h \cdot m, m \otimes h \mapsto m \cdot h, m \mapsto m_{(-1)} \otimes m_{(0)}, m \mapsto m_{<0>} \otimes m_{<1>}$, for all $h \in H, m \in M$), such that M is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e.

$$(h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} = h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}, \quad (1.1)$$

$$(h \cdot m)_{<0>} \otimes (h \cdot m)_{<1>} = h \cdot m_{<0>} \otimes m_{<1>}, \quad (1.2)$$

$$(m \cdot h_2)_{<0>} \otimes h_1 (m \cdot h_2)_{<1>} = m_{<0>} \cdot h_1 \otimes m_{<1>} h_2, \quad (1.3)$$

$$(m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} = m_{(-1)} \otimes m_{(0)} \cdot h. \quad (1.4)$$

The morphisms in $\mathcal{LR}(H)$ are H -bilinear and H -bilinear maps.

If H has a bijective antipode S , $\mathcal{LR}(H)$ becomes a strict braided monoidal category with the following structures: for all $M, N \in \mathcal{LR}(H)$, and $m \in M, n \in N, h \in H$,

$$\begin{aligned} h \cdot (m \otimes n) &= h_1 \cdot m \otimes h_2 \cdot n, & (m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} &= m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}, \\ (m \otimes n) \cdot h &= m \cdot h_1 \otimes n \cdot h_2, & (m \otimes n)_{<0>} \otimes (m \otimes n)_{<1>} &= m_{<0>} \otimes n_{<0>} \otimes m_{<1>} n_{<1>}, \end{aligned}$$

and the braiding

$$c_{M,N} : M \otimes N \mapsto N \otimes M, \quad m \otimes n \mapsto m_{(-1)} \cdot n_{<0>} \otimes m_{(0)} \cdot n_{<1>},$$

and the inverse

$$c_{M,N}^{-1} : N \otimes M \mapsto M \otimes N, \quad n \otimes m \mapsto m_{(0)} \cdot S^{-1}(n_{<1>}) \otimes S^{-1}(m_{(-1)}) \cdot n_{<0>}.$$

2 Main result

In this section, we will give the main result of this paper.

Lemma 2.1. *Let H be a finite dimensional bialgebra. Then we have a functor $F : \mathcal{LR}(H) \longrightarrow {}_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ given for any object $M \in \mathcal{LR}(H)$ and any morphism ϑ by*

$$F(M) = M \quad \text{and} \quad F(\vartheta) = \vartheta,$$

where $H \otimes H^*$ is a bialgebra with tensor product and tensor coproduct.

Proof. For all $M \in \mathcal{LR}(H)$, first of all, define the left action of $H \otimes H^*$ on M by

$$(h \otimes f) \cdot m = \langle f, m_{<1>} \rangle h \cdot m_{<0>}, \quad (2.1)$$

for all $h \in H, f \in H^*$ and $m \in M$. Then M is a left $H \otimes H^*$ -module. Indeed for all $h, h' \in H, f, f' \in H^*$ and $m \in M$,

$$\begin{aligned}
(h \otimes f)(h' \otimes f') \cdot m &= (hh' \otimes ff') \cdot m \\
&= \langle ff', m_{<1>} \rangle hh' \cdot m_{<0>} \\
&= \langle f, m_{<1>} \rangle \langle f', m_{<1>} \rangle h \cdot (h' \cdot m_{<0>}) \\
&= \langle f, m_{<0>} \rangle \langle f', m_{<1>} \rangle h \cdot (h' \cdot m_{<0>} \rangle_{<0>}) \\
&\stackrel{(1.2)}{=} \langle f, (h' \cdot m_{<0>}) \rangle \langle f', m_{<1>} \rangle h \cdot (h' \cdot m_{<0>})_{<0>} \\
&= \langle f', m_{<1>} \rangle (h \otimes f) \cdot (h' \cdot m_{<0>}) \\
&= (h \otimes f) \cdot ((h' \otimes f') \cdot m).
\end{aligned}$$

And

$$(1 \otimes \varepsilon) \cdot m = \langle \varepsilon, m_{<1>} \rangle m_{<0>} = m,$$

as claimed. Next for all $m \in M$, define the left coaction of $H \otimes H^*$ on M by

$$\rho(m) = m_{[-1]} \otimes m_{[0]} = \sum m_{(-1)} \otimes h^i \otimes m_{(0)} \cdot h_i, \quad (2.2)$$

where $\{h_i\}_i$ and $\{h^i\}_i$ are dual bases in H and H^* . Then on one hand,

$$(\Delta_{H \otimes H^*} \otimes id)\rho(m) = \sum m_{(-1)1} \otimes h_1^i \otimes m_{(-1)2} \otimes h_2^i \otimes m_{(0)} \cdot h_i.$$

Evaluating the right side of the equation on $id \otimes g \otimes id \otimes h \otimes id$, we obtain

$$m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.$$

On the other hand

$$\begin{aligned}
(id \otimes \rho)\rho(m) &= \sum m_{(-1)} \otimes h^i \otimes (m_{(0)} \cdot h_i)_{(-1)} \otimes h^j \otimes (m_{(0)} \cdot h_i)_{(0)} \cdot h_j \\
&\stackrel{(1.4)}{=} \sum m_{(-1)} \otimes h^i \otimes m_{(0)(-1)} \otimes h^j \otimes (m_{(0)(0)} \cdot h_i) \cdot h_j \\
&= \sum m_{(-1)1} \otimes h^i \otimes m_{(-1)2} \otimes h^j \otimes m_{(0)} \cdot h_i h_j.
\end{aligned}$$

Evaluating the right side of the equation on $id \otimes g \otimes id \otimes h \otimes id$, we obtain

$$m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.$$

Since $g, h \in H$ were arbitrary, we have

$$(\Delta_{H \otimes H^*} \otimes id)\rho = (id \otimes \rho)\rho.$$

And since

$$(\varepsilon_{H \otimes H^*} \otimes id)(\rho(m)) = \varepsilon(m_{(-1)})m_{(0)} = m,$$

M is a left $H \otimes H^*$ -comodule.

Finally

$$\begin{aligned}
& [(h \otimes f)_1 \cdot m]_{[-1]} (h \otimes f)_2 \otimes [(h \otimes f)_1 \cdot m]_{[0]} \\
&= (h_1 \cdot m_{<0>})_{[-1]} \langle f_1, m_{<1>} \rangle (h_2 \otimes f_2) \otimes (h_1 \cdot m_{<0>})_{[0]} \\
&= \sum \langle f_1, m_{<1>} \rangle ((h_1 \cdot m_{<0>})_{(-1)} h_2 \otimes h^i f_2) \otimes (h_1 \cdot m_{<0>})_{(0)} \cdot h_i \\
&\stackrel{(1.1)}{=} \sum \langle f_1, m_{<1>} \rangle h_1 m_{<0>(-1)} \otimes h^i f_2 \otimes h_2 \cdot m_{<0>(0)} \cdot h_i.
\end{aligned}$$

Evaluating the right side of the equation on $id \otimes g \otimes id$, we obtain

$$\langle f, m_{<1>g_2} \rangle h_1 m_{<0>(-1)} \otimes h_2 \cdot m_{<0>(0)} \cdot g_1.$$

And

$$\begin{aligned}
& (h \otimes f)_1 m_{[-1]} \otimes (h \otimes f)_2 \cdot m_{[0]} \\
&= \sum (h_1 \otimes f_1) (m_{(-1)} \otimes h^i) \otimes (h_2 \otimes f_2) \cdot (m_{(0)} \cdot h_i) \\
&= \sum h_1 m_{(-1)} \otimes f_1 h^i \otimes \langle f_2, (m_{(0)} \cdot h_i)_{<1>} \rangle h_2 \cdot (m_{(0)} \cdot h_i)_{<0>}.
\end{aligned}$$

Evaluating the right side of the equation on $id \otimes g \otimes id$, we obtain

$$\begin{aligned}
& h_1 m_{(-1)} \otimes \langle f, g_1 (m_{(0)} \cdot g_2)_{<1>} \rangle h_2 \cdot (m_{(0)} \cdot g_2)_{<0>} \\
&\stackrel{(1.3)}{=} h_1 m_{(-1)} \otimes \langle f, m_{(0)<1>g_2} \rangle h_2 \cdot m_{(0)<0>} \cdot g_1 \\
&= \langle f, m_{<1>g_2} \rangle h_1 m_{<0>(-1)} \otimes h_2 \cdot m_{<0>(0)} \cdot g_1.
\end{aligned}$$

Therefore M is a left-left Yetter-Drinfeld module over $H \otimes H^*$. It is straightforward to verify that any morphism in $\mathcal{LR}(H)$ is also a morphism in ${}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$. The proof is completed. \square

Lemma 2.2. *Let H be a finite dimensional bialgebra. Then we have a functor $G : {}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD} \longrightarrow \mathcal{LR}(H)$ given for any object $M \in {}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$ and any morphism θ by*

$$G(M) = M \quad \text{and} \quad G(\theta) = \theta.$$

Proof. We denote by ε^* the map ε_{H^*} defined by $\varepsilon_{H^*}(f) = f(1)$ for all $f \in H^*$. For any $M \in {}^{H \otimes H^*}_{H \otimes H^*} \mathcal{YD}$, denote the left $H \otimes H^*$ -coaction on M by

$$m \mapsto m_{[-1]} \otimes m_{[0]},$$

for all $m \in M$. Define the H -bimodule and H -bicomodule structures as follows:

$$h \cdot m = (h \otimes \varepsilon) \cdot m, \quad \rho_L(m) = m_{(-1)} \otimes m_{(0)} = (id \otimes \varepsilon^*)(m_{[-1]}) \otimes m_{[0]}, \quad (2.3)$$

$$m \cdot h = \langle (\varepsilon \otimes id)m_{[-1]}, h \rangle m_{[0]}, \quad \rho_R(m) = m_{<0>} \otimes m_{<1>} = \sum (1 \otimes h^i) \cdot m \otimes h_i. \quad (2.4)$$

for all $h \in H$.

Obviously M is a left H -module. And

$$\begin{aligned} (\Delta \otimes id)\rho_L(m) &= \Delta((id \otimes \varepsilon^*)(m_{[-1]})) \otimes m_{[0]} \\ &= (id \otimes \varepsilon^*)(m_{[-1]1})(id \otimes \varepsilon^*)(m_{[-1]2}) \otimes m_{[0]} \\ &= (id \otimes \varepsilon^*)(m_{[-1]})(id \otimes \varepsilon^*)(m_{[0][-1]}) \otimes m_{[0][0]} \\ &= (id \otimes \rho_L)\rho_L(m). \end{aligned}$$

The counit is straightforward. Thus M is a left H -comodule. For all $h, h' \in M$,

$$\begin{aligned} m \cdot hh' &= \langle (\varepsilon \otimes id)m_{[-1]}, hh' \rangle m_{[0]} \\ &= \langle (\varepsilon \otimes id)m_{[-1]1}, h \rangle \langle (\varepsilon \otimes id)m_{[-1]2}, h' \rangle m_{[0]} \\ &= \langle (\varepsilon \otimes id)m_{[-1]}, h \rangle \langle (\varepsilon \otimes id)m_{[0][-1]}, h' \rangle m_{[0][0]} \\ &= \langle (\varepsilon \otimes id)m_{[-1]}, h \rangle m \cdot h' \\ &= (m \cdot h) \cdot h'. \end{aligned}$$

The unit is obvious. Thus M is a right H -module. Since

$$\begin{aligned} (id \otimes \Delta)\rho_R(m) &= \sum (1 \otimes h^i) \cdot m \otimes h_{i1} \otimes h_{i2} \\ &= \sum (1 \otimes h^i h^j) \cdot m \otimes h^j \otimes h^i \\ &= (\rho_R \otimes id)\rho_R(m), \end{aligned}$$

it follows that M is a right H -comodule. Moreover

$$\begin{aligned} (h \cdot m) \cdot h' &= ((h \otimes \varepsilon) \cdot m) \cdot h' \\ &= \langle (\varepsilon \otimes id)((h \otimes \varepsilon) \cdot m)_{[-1]}, h' \rangle ((h \otimes \varepsilon) \cdot m)_{[0]} \\ &= \langle (\varepsilon \otimes id)[((h_1 \otimes \varepsilon) \cdot m)_{[-1]}(h_2 \otimes \varepsilon)], h' \rangle ((h_1 \otimes \varepsilon) \cdot m)_{[0]} \\ &\stackrel{(1.1)}{=} \langle (\varepsilon \otimes id)((h_1 \otimes \varepsilon)m_{[-1]}), h' \rangle (h_2 \otimes \varepsilon) \cdot m_{[0]} \\ &= \langle (\varepsilon \otimes id)m_{[-1]}, h' \rangle (h \otimes \varepsilon) \cdot m_{[0]} \\ &= h \cdot (m \cdot h'). \end{aligned}$$

Thus M is an H -bimodule. And

$$\begin{aligned} (\rho_L \otimes id)\rho_R(m) &= \sum (id \otimes \varepsilon^*)((1 \otimes h^i) \cdot m)_{[-1]} \otimes ((1 \otimes h^i) \cdot m)_{[0]} \otimes h_i \\ &= \sum (id \otimes \varepsilon^*)((1 \otimes h_1^i) \cdot m)_{[-1]}(1 \otimes h_2^i) \otimes ((1 \otimes h_1^i) \cdot m)_{[0]} \otimes h_i \\ &\stackrel{(1.1)}{=} \sum (id \otimes \varepsilon^*)((1 \otimes h_1^i)m_{[-1]}) \otimes (1 \otimes h_2^i) \cdot m_{[0]} \otimes h_i \end{aligned}$$

$$= (id \otimes \rho_R) \rho_L(m).$$

Thus M is an H -bicomodule.

We now prove (1.1). For all $h \in H, m \in M$,

$$\begin{aligned} & (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} \\ &= ((h_1 \otimes \varepsilon) \cdot m)_{(-1)} h_2 \otimes ((h_1 \otimes \varepsilon) \cdot m)_{(0)} \\ &= (id \otimes \varepsilon^*)(((h_1 \otimes \varepsilon) \cdot m)_{[-1]}(h_2 \otimes \varepsilon)) \otimes ((h_1 \otimes \varepsilon) \cdot m)_{[0]} \\ &\stackrel{(1.1)}{=} (id \otimes \varepsilon^*)((h_1 \otimes \varepsilon)m_{[-1]}) \otimes (h_2 \otimes \varepsilon) \cdot m_{[0]} \\ &= h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}. \end{aligned}$$

We now prove (1.2):

$$\begin{aligned} (h \cdot m)_{<0>} \otimes (h \cdot m)_{<1>} &= ((h \otimes \varepsilon) \cdot m)_{<0>} \otimes ((h \otimes \varepsilon) \cdot m)_{<1>} \\ &= \sum (1 \otimes h^i)(h \otimes \varepsilon) \cdot m \otimes h_i \\ &= \sum (h \otimes \varepsilon)(1 \otimes h^i) \cdot m \otimes h_i \\ &= h \cdot m_{<0>} \otimes m_{<1>}. \end{aligned}$$

We now prove (1.3): On one hand,

$$\begin{aligned} (m \cdot h_2)_{<0>} \otimes h_1(m \cdot h_2)_{<1>} &= \langle (\varepsilon \otimes id)m_{[-1]}, h_2 \rangle m_{[0]<0>} \otimes h_1 m_{[0]<1>} \\ &= \sum \langle (\varepsilon \otimes id)m_{[-1]}, h_2 \rangle (1 \otimes h^i) \cdot m_{[0]} \otimes h_1 h_i. \end{aligned}$$

Evaluating the right side on $id \otimes f$ for all $f \in H^*$, we have

$$\begin{aligned} & \langle (\varepsilon \otimes id)m_{[-1]}, h_2 \rangle (1 \otimes f_2) \cdot m_{[0]} f_1(h_1) \\ &= \langle (\varepsilon \otimes id)(1 \otimes f_1)m_{[-1]}, h \rangle (1 \otimes f_2) \cdot m_{[0]} \\ &\stackrel{(1.1)}{=} \langle (\varepsilon \otimes id)((1 \otimes f_1) \cdot m)_{[-1]}(1 \otimes f_2), h \rangle ((1 \otimes f_1) \cdot m)_{[0]}. \end{aligned}$$

On the other hand,

$$\begin{aligned} m_{<0>} \cdot h_1 \otimes m_{<1>} h_2 &= \sum ((1 \otimes h^i) \cdot m) \cdot h_1 \otimes h_i h_2 \\ &= \sum \langle (\varepsilon \otimes id)((1 \otimes h^i) \cdot m)_{[-1]}, h_1 \rangle ((1 \otimes h^i) \cdot m)_{[0]} \otimes h_i h_2 \end{aligned}$$

Evaluating the right side on $id \otimes f$, we have

$$\begin{aligned} & \langle (\varepsilon \otimes id)((1 \otimes f_1) \cdot m)_{[-1]}, h_1 \rangle ((1 \otimes f_1) \cdot m)_{[0]} f_2(h_2) \\ &= \langle (\varepsilon \otimes id)((1 \otimes f_1) \cdot m)_{[-1]}(1 \otimes f_2), h \rangle ((1 \otimes f_1) \cdot m)_{[0]}. \end{aligned}$$

Hence $(m \cdot h_2)_{<0>} \otimes h_1(m \cdot h_2)_{<1>} = m_{<0>} \cdot h_1 \otimes m_{<1>} h_2$ since f was arbitrary.

We now prove (1.4):

$$\begin{aligned}
& (m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} \\
&= \langle (\varepsilon \otimes id)m_{[-1]}, h \rangle (id \otimes \varepsilon^*)(m_{[0][-1]}) \otimes m_{[0][0]} \\
&= \langle (\varepsilon \otimes id)m_{[-1]1}, h \rangle (id \otimes \varepsilon^*)(m_{[-1]2}) \otimes m_{[0]} \\
&= (id \otimes h)m_{[-1]} \otimes m_{[0]} \\
&= \langle (\varepsilon \otimes id)m_{[-1]2}, h \rangle (id \otimes \varepsilon^*)(m_{[-1]1}) \otimes m_{[0]} \\
&= m_{(-1)} \otimes m_{(0)} \cdot h,
\end{aligned}$$

where in the third equality, $(id \otimes h)m_{[-1]}$ means the second factor of $m_{[-1]}$ acts on h .

Therefore $M \in \mathcal{LR}(H)$. It is straightforward to verify that any morphism in ${}_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ is also a morphism in $\mathcal{LR}(H)$. The proof is completed. \square

Theorem 2.3. *Let H be a finite dimensional bialgebra. Then we have a monoidal category isomorphism*

$$\mathcal{LR}(H) \cong {}_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}.$$

Moreover if H is a Hopf algebra with bijective antipode S , they are isomorphic as braided monoidal categories. Consequently

$$\mathcal{LR}(H) \cong_{D(H \otimes H^*)} \mathcal{M},$$

where $D(H \otimes H^*)$ is the Drinfeld double of $H \otimes H^*$.

Proof. It is easy to see that the functor $F : \mathcal{LR}(H) \rightarrow {}_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ is monoidal and that $F \circ G = id$ and $G \circ F = id$. And for all $M, N \in \mathcal{LR}(H)$, and $m \in M, n \in N$,

$$\begin{aligned}
m_{[-1]} \cdot n \otimes m_{[0]} &\stackrel{(2.2)}{=} \sum (m_{(-1)} \otimes h^i) \cdot n \otimes m_{(0)} \cdot h_i \\
&\stackrel{(2.1)}{=} \sum m_{(-1)} \cdot n_{<0>} \otimes m_{(0)} \cdot n_{<1>}.
\end{aligned}$$

The proof is completed. \square

Corollary 2.4. *(A, H) is an L - R -admissible pair if and only if $(A, H \otimes H^*)$ is an admissible pair.*

By the isomorphism in Theorem 2.3, we can obtain the following result in [5] directly.

Proposition 2.5. *Let H be a finite dimensional Hopf algebra. The canonical braiding of $\mathcal{LR}(H)$ is pseudosymmetric if and only if H is commutative and cocommutative.*

Proof. From [4], the canonical braiding of ${}_{H \otimes H^*}^{H \otimes H^*} \mathcal{YD}$ is pseudosymmetric if and only if $H \otimes H^*$ is commutative and cocommutative. By the bialgebra structure of $H \otimes H^*$, the proof is completed. \square

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References

- [1] C. Kassel, Quantum groups, GTM 155, Springer, Berlin, 1995.
- [2] F. Panaite, F. Van Oystaeyen, L-R-smash product for (quasi-) Hopf algebras, J. Algebra 309(2007): 168–191.
- [3] F. Panaite, F. Van Oystaeyen, L-R-smash biproducts, double biproducts and a braided category of Yetter-Drinfeld-Long bimodules, Rocky Mountain Journal of Mathematics, 40(2008): 2013–2024.
- [4] F. Panaite, M. Staic, F. Van Oystaeyen, Pseudosymmetric braidings, twines and twisted algebras, J. Pure Appl. Algebra, 214(2010): 867–884.
- [5] F. Panaite, M. Staic, More examples of pseudosymmetric braided categories, J. Alg. Appl., 12(2011): 63–75.
- [6] D. E. Radford, The structure of Hopf algebras with a projection, J. Algebra 92(1985): 322–347.
- [7] L. Y. Zhang, L-R smash products for bimodule algebras, Progr. Nat. Science 16(2006): 580–587.